that the group generated by them has a commutator subgroup of order 2, which we shall represent by 1, c.

From the fact that $s_1s_2.s_2s_1$ is of the same order as $s_1^2s_2s_1^2$ and hence also of the same order as s_1s_2 it follows that the order of s_1s_2 is divisible by 4 but not by 8. In fact, $(s_1s_2.s_2s_1)^2=c(s_1s_2)^4$ is of the same order as $(s_1s_2)^2$ only when the order of s_1s_2 is divisible by 4 but not by 8. As $c(s_1s_2)^{-4}=(s_1^2s_2s_1^2)^2=(s_1s_2)^2$ it results that $c=(s_1s_2)^6$; and hence s_1s_2 is of order 4, the order of H being non-divisible by 3. The two conjugates s_1s_2, s_2s_1 must therefore generate the quaternion group. As this includes the inverse of $s_1^2s_2s_1^2$, it is transformed into itself by s_1. Moreover, s_1 has four conjugates under this group. These results may be expressed as follows:

If a group contains only four subgroups of order 3 the operators of this order generate the non-cyclic group of order 9, the alternating group of order 12, or the group of order 24 which does not include a subgroup of order 12. Hence it contains one of these subgroups invariantly.*

§ 2. *Groups which contain just fourteen operators of order three and are generated by these operators.*

If a group contains just fourteen operators of order 3 its Sylow subgroups must be cyclic since 7 is not of the form $1+p+kp^2$.† Hence the seven subgroups of order 3 form a single set of conjugates and are transformed by G according to a transitive group T of degree seven which contains just 7 subgroups of order 3. As in the preceding section there is evidently a (1, 1) correspondence between the operators of order 3 in G and the substitution of this order in T.

Two operators of order 3 in G, s_1, s_2 may be so selected that their product s_1s_2 corresponds to a substitution of order 7 in T. The three conjugates s_1s_2, s_2s_1, $s_1^2s_2s_1^2$ are commutative since T contains only one subgroup of order 7 and s_1s_2 is commutative with every operator of the subgroup H corresponding to the identity in T. From the equations $(s_1s_2.s_2s_1)^7=(s_1s_2)^{14}=(s_1^2s_2s_1^2)^{-7}=(s_1s_2)^{-7}$ it follows that $(s_1s_2)^{21}=1$. As H does not include any operator of order 3, s_1s_2 is of order 7.

The group generated by s_1s_2, s_2s_1 is either of order 7 or of order 49, since these operators are commutative. As this group includes $s_1^2s_2s_1^2$ it is transformed into itself by s_1. If its order is 7 the seven subgroups in question generate the group of order 21 which is defined by its order and the fact that it contains seven subgroups of order 3. As this group is so well known it remains only to prove that s_1s_2, s_2s_1 cannot generate a group of order 48. If this were the case s_1 would transform at least two of its 8 subgroups of order 7 into themselves. As it could not be commutative with the generators of both of these subgroups, it and one of these subgroups would again generate the given group of order 21. Hence it results that, *If a group contains just 14 operators of order 3 these operators generate the semi-metacyclic group of degree 7.*

* *Quarterly Journal of Mathematics*, Vol. 28 (1896), p. 274.
† *Transactions of The American Mathematical Society*, Vol. 6 (1905), p. 58.

INTERPRETATIONS OF THE IDENTICAL RELATIONS BETWEEN THE DETERMINANTS OF AN ARRAY.*

By R. P. BAKER, University of Iowa.

§1. $n=2$, $m=4$

(1) If the elements represent homogeneous coördinates of four points on a line, the 2×2 minors are proportional to the distances between pairs of points, and the identical relation in them gives Ptolemy's theorem.

$$(12)(34)-(13)(24)+(14)(23)=0.$$

The relations lineo-linear in elements and determinants such as,

$$a_1(a_2b_3)+a_2(a_3b_1)+a_3(a_1b_2)=0,$$
$$b_1(a_2b_3)+b_2(a_3b_1)+b_3(a_1b_2)=0,$$

give by linear combination Ptolemy's theorem for the points 1, 2, 3 and any other point in the line, or if this point be at infinity the identical relation between the distances of the three points 1, 2, 3.

(2) The elements represent homogeneous coördinates of two points A, B, in space. The determinants are the line coördinates of AB, and the identical relation the "fundamental relation" between them. The relations

$$z_1(a_2b_3)+z_2(a_3b_1)+z_3(a_1b_2)=0 \quad (z=a,\ b)$$

may be taken as a relation between the distances of any point in the line and the moments of the line with respect to the three planes which pass through a vertex.

If $z_1z_2z_3$ are current coördinates the relation is the equation of a plane through a vertex and the line.

§2. $n=2$, $m=5$.

The general formula gives three independent relations. For any eight elements and their six determinants we have the identity given in the case $n=2$, $m=4$. Thus we have the five relations,

$$A_1;\ (23)(45)-(24)(35)+(25)(34)=0,$$
$$A_2;\ (34)(51)-(35)(41)+(31)(45)=0,$$
$$A_3;\ (45)(12)-(41)(52)+(42)(51)=0,$$
$$A_4;\ (51)(23)-(52)(13)+(53)(12)=0,$$
$$A_5;\ (12)(34)-(13)(24)+(14)(23)=0.$$

*Sequel to a paper with similar title in the January number of the MONTHLY.

These relations are dependent in any case on some three of them, but the same three cannot always be taken as independent. There exist, in fact, five identities in the determinants:

$$A'_1\,;\ (12)A_2+(13)A_3-(41)A_4-(51)A_5=0,$$
$$A'_2\,;\ (23)A_3+(24)A_4-(52)A_5-(12)A_1=0,$$
$$A'_3\,;\ (34)A_4+(35)A_5-(13)A_1-(23)A_2=0,$$
$$A'_4\,;\ (45)A_5+(41)A_1-(24)A_2-(34)A_3=0,$$
$$A'_5\,;\ (51)A_1+(52)A_2-(35)A_3-(45)A_4=0.$$

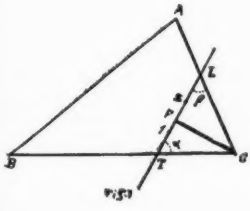
The trivial case of all vanishing determinants being excluded, suppose that (12) is not zero. Then if A_3, A_4, A_5, vanish the relations A'_1, A'_2, show that A_1, A_2, also vanish, but the vanishing of any other three of the A's will not necessarily entail the vanishing of the remaining pair. There are also twenty lineo-linear relations.

The geometrical interpretation is similar to that in the case of a 2×4 matrix.

Trilinear Coordinates.

If the fundamental triangle be ABC and a_1 is the perpendicular distance from the point 1 to the side BC, to interpret $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ drop a perpendicular from C to the line 12 and let 12 meet BC at the angle α, AC at the angle β.

Then $1T=a_1/\sin\alpha$, $\quad 2T=a_2/\sin\alpha$,
$\quad 1L=b_1/\sin\beta$, $\quad 2L=b_2/\sin\beta$,



$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \sin\alpha\,\sin\beta \begin{vmatrix} 1T & 1L \\ 2T & 2L \end{vmatrix} = \sin\alpha\,\sin\beta \begin{vmatrix} 1T & TL \\ 2T & TL \end{vmatrix}$$
$$= TL\,\sin\alpha\,\sin\beta\,\overline{(12)}.$$

Now $TL=TF-FL=CF(\cot\alpha+\cot\beta)$.

$$\therefore \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = CF\sin C\,\overline{(12)} = (\text{moment of line 12 with respect to } C).\sin C.$$

To interpret $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, we have the identities,

$$BC.a_1+CA.b_1+AB.c_1=2\triangle,$$
$$BC.a_2+CA.b_2+AB.c_2=2\triangle,$$
$$BC.a_3+CA.b_3+AB.c_3=2\triangle,$$

whence $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \dfrac{2.\triangle}{BC} \begin{vmatrix} b_1 & c_1 & 1 \\ b_2 & c_2 & 1 \\ b_3 & c_3 & 1 \end{vmatrix} = \dfrac{2.\triangle}{BC}[(b_1c_2)+(b_2c_3)+(b_3c_1)]$

$$=\frac{2.\triangle}{BC}[\overline{\overline{12}}.\sin A+\overline{\overline{23}}.\sin A+\overline{\overline{31}}.\sin A],$$

where $\overline{\overline{12}}$ denotes the moment of the line with respect to A. That is

$$(a_1\, b_2\, c_3)=\frac{2.\triangle\ \sin A}{BC}\ \text{Area } 123=\frac{4.\triangle^2}{AB.BC.CA}\ \text{Area } 123.$$

§3. $n=3$, $m=4$.

There are no relations between the determinants; there are three relations lineo-linear in the elements and determinants and twelve relations lineo-linear in the determinants and their 2×2 minors. To interpret in R_2,

(1) The lineo-linear relations between elements and determinants

$$x_1(a_2b_3c_4)-x_2(a_3b_4c_1)+x_3(a_4b_1c_2)-x_4(a_1b_2c_3)=0\quad[x=(a,\ b,\ c)].$$

The determinants are proportional to the areas of the triangles, the elements to the distances of the points from the sides of the reference triangle.

The relation being homogeneous in rows and columns proportionality factors need not be considered.

Making the usual convention as to signs of triangular areas, we have the theorem:

"In a complete quadrangle the sum of the products of the areas of the triangles into the distances of the remaining vertices from any line is zero."

If the line is at infinity this reduces to the areal identity $234.-341.+412.-123.=0$.

(2) The relation,

$$(a_1b_2)(a_3b_4c_1)-(a_1b_3)(a_2b_4c_1)+(a_1b_4)(a_2b_3c_1)=0.$$

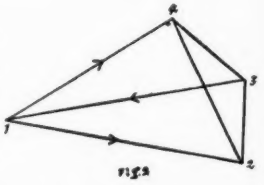
Proportionality factors being neglected as before, (a_1b_2) is the moment of the line 12 with respect to C multiplied by $\sin C$. Also $(a_1b_2c_3)$ is proportional to the area of the triangle 123.

The relation gives,

Moment $12\times$Area$(341)-$Moment $13\times$Area$(241)+$Moment $14\times$Area$(231)=0$,

which may be interpreted as follows:

"If three concurrent forces 12, 13, 14, be proportional to the length of the lines 12, 13, 14, and to the areas of the triangles 341, 421, 231 jointly and have their signs determined by the directions in which the triangles are described they are in equilibrium."



In the case of 1234 being a parallelogram this reduces to the parallelogram of forces.

Or, expressing the areas of the triangles by the product of adjacent sides and sine of included angle, and the moments by the product of lengths of line and perpendicular, and expressing the perpendicular by the product of the line $C1$ and the sine of the proper angle we have, if angle $C12=P$, $213=Q$, $314=R$, and the line factors, now all common, be divided out,

$$\sin(P+Q+R)\sin Q-\sin(P+Q)\sin(Q+R)+\sin P\sin R=0,$$

a trigonometrical identity which reduces to the various forms of the addition theorem when $Q=\frac{1}{2}\pi$, $P=\frac{1}{2}\pi$.

NOTE ON FINDING THE COMPLEMENTARY FUNCTION OF A LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS WHEN THE AUXILIARY EQUATION HAS EQUAL ROOTS.

By B. F. FINKEL.

The following method, which is the one virtually implied though not expressly stated in Forsyth's *Differential Equations*, Art. 44, p. 64, has the advantage of simplicity and elegance of presentation, and is worthy, therefore, of being better known. Dr. G. E. Fisher, of the University of Pennsylvania, has used the method for some years in his classes. Dr. Fisher does not claim that the method is original with him, neither does he remember from whence the suggestion came to him.

As I am not aware that it has been published elsewhere, at least in none of the text books on the subject that I have examined, it is thought that its appearance in the MONTHLY will be helpful to many of its readers.

Let $y=Ae^{ax}+Be^{\beta x}+\ldots\ldots+Le^{lx}$ be the *Complementary Function* of the general linear differential equation

$$\frac{d^ny}{dx^n}+A_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots+A_{n-1}\frac{dy}{dx}+A_ny=V.$$

Now if two of the roots are equal, as, for example, $\alpha=\beta$, we have

$$y=(A+B)e^{ax}+Ce^{\gamma x}+\ldots\ldots+Le^{lx}=A_1e^{ax}+Ce^{\gamma x}+\ldots\ldots+Le^{lx}.*$$

We thus lack an arbitrary constant, there now being only $n-1$. In order to obtain the complete primitive, let us suppose, for the moment, that α and β are different, and that $\beta-\alpha=h$. A particular solution would then be

*Forsyth's *Differential Equations*, 3rd edition, p. 64.

$$y=Ae^{\beta x}+Be^{\alpha x}=Ae^{(\alpha+h)x}+Be^{\alpha x},$$

for all values of A and B.

If $A=-B=\frac{1}{h}$, $y=\frac{1}{h}(e^{(\alpha+h)x}-e^{\alpha x})$ would still be a particular integral.

When $h\doteq 0$, $\beta\doteq\alpha$, and $y=\frac{1}{h}(e^{(x+h)x}-e^{\alpha x})$ becomes a particular solution for the case of equal roots, that is,

$$y=\operatorname*{Lim}_{h\doteq 0}\left[\frac{e^{(\alpha+h)x}-e^{\alpha x}}{h}\right]=\frac{d}{d\alpha}[e^{\alpha x}]=xe^{\alpha x}$$

is a particular solution when $\alpha=\beta$.

Hence, when two roots are equal, in order to find a second particular integral, we need only to obtain the derivative with respect to one of the equal roots, of the first particular integral.

Thus, if $\alpha=\beta$ and if $y=e^{\alpha x}$ is a particular integral, a second particular integral is $y=\frac{d}{d\alpha}[e^{\alpha x}]=xe^{\alpha x}$.

In the same way, we find a third particular integral, if three roots are equal. Thus, suppose $\alpha=\beta=\gamma$, and that $e^{\alpha x}$, $xe^{\alpha x}$ are two particular integrals. If we assume, for the moment, that β and γ are different and that $\gamma-\beta=k$, we would have as a particular integral $y=Cxe^{\gamma x}+Dxe^{\beta x}$, for all values of C and D.

If $C=-D=\frac{1}{k}$, $y=\frac{1}{k}(xe^{(\beta+k)x}-xe^{\beta x})$ would still be a particular integral.

When $k\doteq 0$, $\gamma\doteq\beta$, and $y=\frac{1}{k}(xe^{(\beta+k)x}-xe^{\beta x})$ becomes a particular integral for the case when $\beta=\gamma$. But

$$y=\operatorname*{Lim}_{k\doteq 0}\left[\frac{xe^{(\beta+k)x}-xe^{\beta x}}{h}\right]=\frac{d}{d\beta}[xe^{\beta x}]=x^2e^{\beta x}.$$

Hence, if $\alpha=\beta=\gamma$, three particular integrals are $e^{\alpha x}$, $xe^{\alpha x}$, and $x^2e^{\alpha x}$.

In the same way we find r particular integrals when r roots are equal. In the case of r equal roots, the *complementary function* is

$$y=A_1e^{\alpha x}+A_2\frac{d}{dx}(e^{\alpha x})+A_3\frac{d^2}{dx^2}(e^{\alpha x})+A_4\frac{d^3}{dx^3}(e^{\alpha x})+\cdots\cdots$$

$$+A_r\frac{d^{r-1}}{dx^{r-1}}(e^{\alpha x})+\cdots\cdots+Le^{lx}.$$

Forsyth uses this method in solving example 6, p. 68, though he does not expressly state the method.

MULTIPLY PERFECT ODD NUMBERS WITH THREE PRIME FACTORS.

By R. D. CARMICHAEL.

The object of this note is to prove the following

PROPOSITION. There exist no multiply perfect odd numbers containing only three primes.

Let the numbers here considered be of the form $p_1^{a_1} p_2^{a_2} p_3^{a_3}$, where p_1, p_2, p_3 are distinct odd primes, and $p_1<p_2<p_3$. And also let the multiplicity be m, where $m>1$. Now, by definition, the multiplicity times the number equals the sum of the factors. Hence,

$$(1) \qquad mp_1^{a_1} p_2^{a_2} p_3^{a_3} = \frac{p_1^{a_1+1}-1}{p_1-1}\cdot\frac{p_2^{a_2+1}-1}{p_2-1}\cdot\frac{p_3^{a_3+1}-1}{p_3-1}.$$

$$(2) \qquad m = \frac{p_1^{a_1+1}-1}{p_1^{a_1}(p_1-1)}\cdot\frac{p_2^{a_2+1}-1}{p_2^{a_2}(p_2-1)}\cdot\frac{p_3^{a_3+1}-1}{p_3^{a_3}(p_3-1)}.$$

$$(3) \qquad m < \frac{p_1}{p_1-1}\cdot\frac{p_2}{p_2-1}\cdot\frac{p_3}{p_3-1}.$$

The right member of (3) is greatest when the primes are smallest. By substituting 3, 5, 7 for p_1, p_2, p_3, we find that $m\leqq 2$. Now, with $m=2$ in (3), it is easily shown that we must have $p_1=3$, $p_2=5$, $p_3<16$, whence $p_3=7$, 11, or 13. Then (1) becomes

$$(4) \qquad 2^4.3^{a_1}.5^{a_2}.p_3^{a_3}(p_3-1)=(3^{a_1+1}-1)(5^{a_2+1}-1)(p_3^{a_3+1}-1).$$

When $p_3=7$, equation (4) becomes

$$(5) \qquad 2^5.3^{a_1+1}.5^{a_2}.7^{a_3}=(3^{a_1+1}-1)(5^{a_2+1}-1)(7^{a_3+1}-1).$$

If a_3+1 is even, $7^{a_3+1}-1$ is divisible by 7^2-1, which contains 2^4. But, in any event, $5^{a_2+1}-1$ contains 2^2 and $3^{a_2+1}-1$ contains 2. The right member will then contain 2^7, which is impossible. Hence, a_3+1 is odd. Since odd powers of 7 end in 7 or 3, $7^{a_3+1}-1$ is not divisible by 5. This requires a_1+1 to be even; as odd powers of 3 end in 3 or 7, and $3^{a_1+1}-1$ is therefore not then divisible by 5. Since a_1+1 is even, $3^{a_1+1}-1$ is divisible by $3^2-1=2^3$. The right member of (5) then contains 2^6. This is impossible. Hence, there are no numbers of the type here considered when $p_3=7$.

Next, for $p_3=11$, (4) becomes

$$(6) \qquad 2^5.3^{a_1}.5^{a_2+1}.11^{a_3}=(3^{a_1+1}-1)(5^{a_2+1}-1)(11^{a_3+1}-1).$$

We can here, as above, show that a_3+1 is odd. Likewise, that a_1+1 is odd.

Then $3^{a_1+1}-1$ does not contain 5. Hence, the right member contains the factor 5 only in $11^{a_2+1}-1$. But this factor must occur at least twice, as $a_2 \neq 0$. By writing $(10+1)^{a_2+1}-1$ and expanding, we may easily show that it contains 5 only once unless a_3+1 is divisible by 5. Then, let $a_3+1=5n$. Now, $11^{5n}-1$ is divisible by 11^5-1, which contains a prime greater than 11. Hence, $p_3=11$ yields no numbers of the type here considered.

Finally, for $p_3=13$, (4) becomes

$$2^6.3^{a_1+1}.5^{a_2}.13^{a_3}=(3^{a_1+1}-1)(5^{a_2+1}-1)(13^{a_3+1}-1). \tag{7}$$

If a_3+1 is even, $13^{a_3+1}-1$ is divisible by 13^2-1. This introduces the inadmissible factor 7. Hence, a_3+1 is odd. The odd powers of 13 end in 3 or 7. Hence, $13^{a_3+1}-1$ is not now divisible by 5. If a_1+1 is odd, $3^{a_1+1}-1$ is not divisible by 5. But to satisfy the equation, it must contain 5. Hence, a_1+1 is even, and $3^{a_1+1}-1$ then contains the factor $3^2-1=2^3$. $5^{a_2+1}-1$ always contains 2^2, and $13^{a_3+1}-1$ always contains 2^2. Hence, the right member contains 2^7, which is impossible. Therefore, this case yields no numbers of the type here considered.

DEPARTMENTS.

ALGEBRA.

250. Proposed by **PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.**

Factor $a^2b^2(x^2+y^2)(a^2y^2+b^2x^2-a^2b^2)=(a^4y^2+b^4x^2)[\sqrt{(a^2y^2+b^2x^2)}+ab]^2$.

Solution by the PROPOSER.

Let $x=r\cos\theta$..... (1), $y=r\sin\theta$..... (2); then the given expression equated to zero becomes

$$(b^2\cos^2\theta+a^2\sin^2\theta)[a^2b^2-(b^4\cos^2\theta+a^4\cos^2\theta)]r^2$$
$$-2ab\sqrt{[(b^2\cos^2\theta+a^2\sin^2\theta)](b^4\cos^2\theta+a^4\sin^2\theta)}r$$
$$=a^2b^2(b^4\cos^2\theta+a^4\sin^2\theta+a^2b^2) \text{.....} (3).$$

Multiplying both sides of (3) by the coefficient of r^2 and noticing that

$$a^2b^2-(b^4\cos^2\theta+a^4\sin^2\theta)=a^2b^2(\sin^2\theta+\cos^2\theta)-(b^4\cos^2\theta+a^4\sin^2\theta)$$
$$=b^2(a^2-b^2)\cos^2\theta+a^2(a^2-b^2)\sin^2\theta=(a^2-b^2)(b^2\cos^2\theta+a^2\sin^2\theta) \text{.....} (4),$$

and similarly,

$$a^2b^2+b^4\cos^2\theta+a^4\sin^2\theta=(a^2+b^2)(b^2\cos^2\theta+a^2\sin^2\theta) \text{.....} (5).$$

Completing the square, using the positive sign of the radical,

$$r(a^2-b^2)(b^2\cos^2\theta-a^2\sin^2\theta)=ab(a^2+b^2)\sqrt{(b^2\cos^2\theta+a^2\sin^2\theta)}\ldots\ldots(6).$$

Multiplying both sides of (6) by $r(b^2\cos^2\theta-a^2\sin^2\theta)$, squaring, and putting in the values from (1) and (2),

$$(a^2-b^2)^2(b^2x^2-a^2y^2)^2-a^2b^2(a^2+b^2)^2(b^2x^2+a^2y^2)=0\ldots\ldots(7).$$

This is one factor of the given expression. Using the negative sign after completing the square in (3), and employing (4) and (5),

$$(a^2-b^2)\ (b^2\cos^2\theta-a^2\sin^2\theta)r\sqrt{(b^2\cos^2\theta+a^2\sin^2\theta)}\ [r\sqrt{(b^2\cos^2\theta+a^2\sin^2\theta)}+ab]$$
$$=0\ldots\ldots(8).$$

Equating the last factor to zero, rationalizing, using (1) and (2), we have $a^2y^2+b^2x^2-a^2b^2$ as a second factor.

Also solved by G. B. M. Zerr.

251. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\frac{1}{n+1}+\frac{1}{2(n+2)}+\frac{1}{3(n+3)}+$ etc.,$=$

$$\frac{1}{n^2}+\tfrac{1}{2}\left[\frac{1}{n-1}+\frac{1}{2(n-2)}+\frac{1}{3(n-3)}+\ldots\ldots+\frac{1}{l(n-l)}\right],$$

l being equal to $n-1$, n being any positive integer greater than one.

Solution by L. E. NEWCOMB, Los Gatos, Cal.

The general term is, $\frac{1}{r(n+r)}=\frac{1}{nr}-\frac{1}{nr(r+n)}$. Let $r=1, 2, 3, \ldots\ldots$ in succession; then $\frac{1}{n+1}=\frac{1}{n}-\frac{1}{n(n+1)}$, $\frac{1}{2(n+2)}=\frac{1}{2n}-\frac{1}{n(n+2)}$, $\frac{1}{3(n+3)}=\frac{1}{3n}-\frac{1}{n(n+3)}$.

$\therefore$ Sum $=\frac{1}{n}-\frac{1}{n(n+1)}+\frac{1}{2n}-\frac{1}{n(n+2)}+\frac{1}{3n}-\frac{1}{n(n+3)}\ldots\ldots$ and all the terms after the rth vanish.

$$\therefore \text{Sum}=\frac{1}{n}+\frac{1}{2n}+\frac{1}{3n}+\ldots\ldots+\frac{1}{n^2}\equiv\frac{1}{n^2}+\left[\frac{1}{n}+\frac{1}{2n}+\frac{1}{3n}+\ldots\ldots+\frac{1}{ln}\right](1).$$

In the series (2) $\frac{1}{n-1}+\frac{1}{2(n-2)}+\frac{1}{3(n-3)}+\ldots\ldots+\frac{1}{l(n-l)}$, the general

term is $\frac{1}{r(n-r)}=\frac{1}{nr}+\frac{1}{n(n-r)}$, and since $\frac{1}{n-1}=\frac{1}{n}+\frac{1}{n(n-1)}$, $\frac{1}{2(n-2)}=\frac{1}{2n}+\frac{1}{n(n-2)}$, $\frac{1}{l(n-l)}=\frac{1}{ln}+\frac{1}{n(n-l)}\equiv\frac{1}{n(n-1)}+\frac{1}{n}$, the sum of (2)$=\frac{1}{n}+\frac{1}{2n}+\frac{1}{3n}\ldots\ldots\ldots\frac{1}{n}+[\frac{1}{n(n-1)}+\frac{1}{n(n-2)}+\ldots.+\frac{1}{n}]$. The terms within and without the parenthesis are now plainly identical; consequently, $\frac{1}{2}[\frac{1}{n-1}+\frac{1}{2(n-2)}+\frac{1}{3(n-3)}+\ldots.+\frac{1}{l(n-l)}]$ substituted for $\frac{1}{n}+\frac{1}{2n}+\frac{1}{3n}\ldots.$ $\ldots.\frac{1}{ln}$ in the right hand member of (1) will satisfy the equation.

II. Solution by R. D. CARMICHAEL, Hartselle, Alabama.

Represent the series of the first member by S_n. Then,

$$S_2=\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}\ldots.=\tfrac{1}{2}[(1-\tfrac{1}{3})+(\tfrac{1}{2}-\tfrac{1}{4})+(\tfrac{1}{3}-\tfrac{1}{5})+(\tfrac{1}{4}-\tfrac{1}{6})\ldots.]=\tfrac{1}{2}(1+\tfrac{1}{2}).$$

$$S_3=\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\ldots.=\tfrac{1}{3}[(1-\tfrac{1}{4})+(\tfrac{1}{2}-\tfrac{1}{5})+(\tfrac{1}{3}-\tfrac{1}{6})+(\tfrac{1}{4}-\tfrac{1}{7})\ldots.]=\tfrac{1}{3}(1+\tfrac{1}{2}+\tfrac{1}{3}).$$

$$S_n=\frac{1}{n}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}\ldots.+\frac{1}{n})=\frac{1}{n^2}+\frac{1}{n}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}\ldots.+\frac{1}{n-1})$$

$$=\frac{1}{n^2}+\frac{1}{n}(\frac{1}{n-1}+\frac{1}{n-2}+\frac{1}{n-3}\ldots.+\tfrac{1}{3}+\tfrac{1}{2}+1)$$

$$\therefore 2S_n=\frac{2}{n^2}+\frac{1}{n}[(1+\frac{1}{n-1})+(\tfrac{1}{2}+\frac{1}{n-2})+(\tfrac{1}{3}+\frac{1}{n-3})\ldots.+(1+\frac{1}{n-1})].$$

$$\therefore S_n=\frac{1}{n^2}+\tfrac{1}{2}[\frac{1}{n-1}+\frac{1}{2(n-2)}+\frac{1}{3(n-3)}+\ldots\ldots\ldots\frac{1}{l(n-l)}],\ l \text{ being equal to } n-1.$$

Also solved by G. W. Greenwood, Henry Heaton, and G. B. M. Zerr.

252. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Solve (1) $x-y=\frac{1}{3}\pi$; (2) $\sin x=\cos^3 y$.

Solution by J. SCHEFFER, Hagerstown, Md.

Since $x=y+\frac{1}{3}\pi$, we have $\sin x=\frac{1}{2}\sin y+\frac{1}{2}\sqrt{3}\cos y$.

$\therefore \frac{1}{2}\sin y+\frac{1}{2}\sqrt{3}\cos y=\cos^3 y$.

$\therefore 1-\cos^2 y=4\cos^6 y-4\sqrt{3}\cos^4 y+3\cos^2 y$, or $4\cos^6 y-4\sqrt{3}\cos^4 y+4\cos^2 y-1=0$; putting $\cos^2 y=z$, we get $z^3-\sqrt{3}z^2+z-\frac{1}{4}=0$; putting $z=t+\frac{1}{3}\sqrt{3}$, we get $t^3=\frac{1}{4}-\frac{1}{9}\sqrt{3}$; $\therefore t=(\frac{1}{4}-\frac{1}{9}\sqrt{3})^{\frac{1}{3}}$.

$\therefore y=\cos^{-1}\pm[\frac{1}{3}\sqrt{3}+(\frac{1}{4}-\frac{1}{9}\sqrt{3})^{\frac{1}{2}}]^{\frac{1}{2}}$, $x=\sin^{-1}\pm[\frac{1}{3}\sqrt{3}+(\frac{1}{4}-\frac{1}{9}\sqrt{3})^{\frac{1}{2}}]^{\frac{1}{2}}$.

The two acute angles x and y are very nearly* 71° 1′ and 11° 1′.

Also solved by R. D. Carmichael, S. A. Corey, Henry Heaton, G. B. M. Zerr.

AVERAGE AND PROBABILITY.

174. Proposed by HENRY HEATON, Atlantic, Iowa.

Chords are drawn through every point of the surface of a given circle in every possible direction. What is their average length?

Solution by J. EDWARD SANDERS, Reinersville, Ohio.

Let θ=the angle the chord makes with r, then its length is $2\sqrt{(a^2-r^2\sin^2\theta)}$.

$$\therefore \Delta=\frac{4}{\pi a^2}\int_0^{\frac{1}{2}\pi}d\theta\int_0^a 2\sqrt{(a^2-r^2\sin^2\theta)}rdr=\frac{8a}{3\pi}\int_0^{\frac{1}{2}\pi}\frac{1-\cos^3\theta}{\sin^2\theta}d\theta=\frac{16a}{3\pi}.$$

Also solved by G. B. M. Zerr, and the Proposer.

175. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

If a line l is divided into three parts by two points taken at random on it, what is the mean value of the triangle whose sides are equal to the three parts? (Only those cases are to be considered in which the the three parts will form a triangle.)

Remark by S. A. COREY, Hiteman, Iowa.

Assuming that by mean value is meant average area, the problem becomes identical with problem 156, Average and Probability, a solution of which appeared in the MONTHLY in December, 1904, on page 237. The average area as there given is† $\pi l^2/105$.

CALCULUS.

213. Proposed by EDWIN L. RICH, Schenectady, N. Y.

Let $f(x)$ be any function of x, and $f'(x)$ its derivative. If $u=[f'(x)]^{-\frac{1}{2}}$, $v=f(x)[f'(x)]^{-\frac{1}{2}}$, then $\frac{1}{u}\frac{d^2u}{dx^2}-\frac{1}{v}\frac{d^2v}{dx^2}=0$.

I. Solution by W. L. TRYON, Ithaca, N. Y.

If $u=[f'(x)]^{-\frac{1}{2}}$, $\frac{du}{dx}=-\frac{1}{2}[f'(x)]^{-\frac{3}{2}}f''(x)$,

*By the use of graphical methods Dr. Westlund obtains $x=71°\ 1'\ 27''$, $y=11°\ 1'\ 27''$. Mr. Heaton's result, correct to the sixth decimal place, is $x=71°\ 1'\ 28''$, $y=11°\ 1'\ 28''$. G.

†Solutions were contributed by Henry Heaton, J. Edward Sanders, and G. B. M. Zerr. The problem is solved in Williamson's *Integral Calculus*, Seventh Edition, p. 359. G.

$$\frac{d^2u}{dx^2}=-\tfrac{1}{2}\{-\tfrac{3}{2}[f'(x)]^{-\frac{5}{2}}[f''(x)]^2+[f'(x)]^{-\frac{3}{2}}[f'''(x)]\},$$

$$\frac{1}{u}\frac{d^2u}{dx^2}=(-\tfrac{1}{2})\{-\tfrac{3}{2}[f'(x)]^{-2}[f''(x)]^2+[f'(x)]^{-1}[f'''(x)]\}.$$

Also, if $v=f(x)[f'(x)]^{-\frac{1}{2}}$, $\frac{dv}{dx}=[f'(x)]^{\frac{1}{2}}-\frac{1}{2}f(x)[f'(x)]^{-\frac{3}{2}}f''(x)$,

$$\frac{d^2v}{dx^2}=-\tfrac{1}{2}f(x)\{-\tfrac{3}{2}[f(x)]^{-\frac{5}{2}}[f''(x)]^2+[f'(x)]^{-\frac{3}{2}}[f'''(x)]\},$$

$$\frac{1}{v}\frac{d^2v}{dx^2}=-\tfrac{1}{2}\{-\tfrac{3}{2}[f(x)]^{-2}[f''(x)]^2+[f'(x)]^{-1}[f'''(x)]\}=\frac{1}{u}\frac{d^2u}{dx^2}.$$

$$\therefore \frac{1}{u}\frac{d^2u}{dx^2}-\frac{1}{v}\frac{d^2v}{dx^2}=0.$$

II. Solution by W. W. LANDIS, Dickinson College, Carlisle, Pa.

Since $u=(y')^{-\frac{1}{2}}$, $v=y(y')^{-\frac{1}{2}}$, where $y'=f'(x)$. $v=uy$.

$$\therefore v''=\frac{d^2v}{dx^2}=uy''+2u'y'+yu''. \quad \therefore \frac{v''}{v}=\frac{y''}{y}+\frac{2u'y'}{uy}+\frac{u''}{u}.$$

$$\therefore \frac{u''}{u}-\frac{v''}{v}=-\left[\frac{y''}{y}+\frac{2u'y'}{uy}\right]=0.$$

$\therefore uy''=-2u'y'$, but $u'=-\frac{1}{2}(y')^{-\frac{3}{2}}y''$. $\therefore -2u'y'=(y')^{-\frac{1}{2}}y''=uy''$. This proves the required result.

Also solved by F. Anderegg, R. D. Carmichael, S. A. Corey, J. Scheffer, and G. B. M. Zerr.

214. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Prove that $\pi^2=6.\frac{2^2}{2^2-1}.\frac{3^2}{3^2-1}.\frac{5^2}{5^2-1}\cdots\cdots$ where the squared numbers in the numerator are the natural *primes* in order.

Solution by WILLIAM HOOVER, Athens, Ohio.

We have from trigonometry, $\pi^2=6(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots)\ldots\ldots\ldots(1)$.

In a foot-note to page 109, Boole's *Finite Differences*, edition 1872, is the following:

$$\text{“}S\equiv 1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\ldots\ldots;$$

$$\frac{1}{2^{2n}}S\equiv\frac{1}{2^{2n}}+\frac{1}{4^{2n}}+\ldots\ldots;$$

$$\left(1-\frac{1}{2^{2n}}\right)S=1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\ldots;$$

$$\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)S=1+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\ldots;$$

....

all of the terms of the form $1/(2p)^{2n}$ being removed.

Ultimately, $\left[\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)\left(1-\frac{1}{5^{2n}}\right)\ldots\ldots\ldots\right]S=1.$"

Let $S=S_1$ when $n=1$; then equation (1) becomes

$$\pi^2=6S_1=6\,\frac{1}{1-\frac{1}{2^2}}\cdot\frac{1}{1-\frac{1}{3^2}}\cdot\frac{1}{1-\frac{1}{5^2}}\ldots\ldots=6\frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\ldots\ldots\ldots$$

It is plain into what general form all of Bernoulli's Numbers of the form given by Boole, viz:

$$B_{2n-1}=\frac{2(2n)!}{(2\pi)^{2n}}\left[1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\ldots\right],$$

can be thrown.

Also solved by F. Anderegg, and G. B. M. Zerr.

GEOMETRY.

276. Proposed by G. I. HOPKINS, Manchester, N. H.

ABC is an equilateral triangle whose vertices are the centers of circles with radius AB, and H is the center of the arc AB. From F, the point of intersection of the circles whose centers are A and C, a line is drawn through H to the circumference CAN. Draw BN, and prove that the angle ABN is an angle of a regular pentagon.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $BN=AB=AC=BC=a$. Then $BF=a\sqrt{3}$, $\angle ABF=\pi/6$, $\angle HFB=\pi/12$. $\therefore \angle HDB=\pi/4$.

$BN : BF=\sin\pi/6 : \sin N$; $a : a\sqrt{3}=\sin\pi/6 : \sin N$.

$\therefore \sin N=\sqrt{3}\sin\pi/6=\frac{3-\sqrt{3}}{2\sqrt{2}}=.44828.$

$\therefore N=26^\circ\ 38'$. $\therefore ABN=108^\circ\ 22'$, or $22'$ larger than the angle of a regular pentagon. The construction thus gives merely a rough approximation.

Also solved by L. E. Newcomb, and J. Scheffer.

277. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

It is tacitly assumed in elementary geometry that as the number of sides of a regular polygon inscribed in a circle is increased, *in any manner*, that its perimeter has a fixed limit. Beginning with a square and then continually doubling the number of sides we get for the perimeter $2^{n+2}\sqrt{[2-E^n(0)]}$, where $E(x)\equiv\sqrt{(2+x)}$. Beginning with a hexagon we get $2^{m+1}3\sqrt{[2-E^m(1)]}$. The definition of the length of a circle assumes that these expressions have the same limit as $n\doteq\infty$ and $m\doteq\infty$. Prove it.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Putting $\cos x=b$, we have $\sin\frac{1}{2}x=\sqrt{[\frac{1}{2}(1-b)]}=\frac{1}{2}\sqrt{[2-2b]}$, $\cos\frac{1}{2}x=\frac{1}{2}\sqrt{[2+2b]}$, $\sin\frac{1}{4}x=\frac{1}{2}\sqrt{[2-\sqrt{(2+2b)}]}$, $\cos\frac{1}{4}x=\frac{1}{2}\sqrt{[2+\sqrt{(2+2b)}]}$, $\sin\frac{1}{8}x=\frac{1}{2}\sqrt{\{2-\sqrt{[2+\sqrt{(2+2b)}]}\}}$, etc. Let the radius of a circle=1, then we may represent the perimeter of a regular polygon by $2^{n+2}\sin\frac{1}{2^n}x$. For $n=\infty$, this assumes the indeterminate form $\infty\doteq 0$; but we may reduce it to $\frac{2^2\sin(1/2^n)x}{1/2^n}$. Differentiating numerator and denominator with reference to n, we get $2^2x\cos(1/2^n)x$. For $n=\infty$, this becomes 2^2x. For $x=90°$, $2^2\times 90=360°$.

In regard to the hexagon, we have similarly, $\frac{3\sin(1/2^n)\times 120°}{1/2^n}$, which after differentiation as above, becomes $3\times 120°=360°$, so that the limit in both cases is 360°.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$E^n(0)=\sqrt{\{2+\sqrt{[2+\sqrt{(2+\sqrt{(2+}}}}$, etc.))]};

$E^m(1)=\sqrt{\{2+\sqrt{[2+\sqrt{(2+\sqrt{(2+}}}}$, etc.))]}, the last term in the root being $\sqrt{3}$.

Let $E^n(0)=E^m(1)=x$. $\therefore\sqrt{(2+x)}=x$, or $x=2$.

$$\frac{2^{n+2}\sqrt{[2-E^n(0)]}}{2^{m+1}3\sqrt{[2-E^m(1)]}}=\frac{2^{n+2}\sqrt{[2-x]}}{2^{m+1}\sqrt{[18-9x]}}=\frac{2^{n+1}\sqrt{[8-4x]}}{2^{m+1}\sqrt{18-9x]}}.$$

$$\frac{2^{n+1}}{2^{m+1}}\frac{\sqrt{[8-4x]}}{\sqrt{[18-9x]}}=\frac{\sqrt{[8-4x]}}{\sqrt{[18-9x]}},\text{ when } n=m=\infty.$$

If $\frac{\sqrt{[8-4x]}}{\sqrt{[18-9x]}}=1$, $8-4x=18-9x$, or $x=2$,

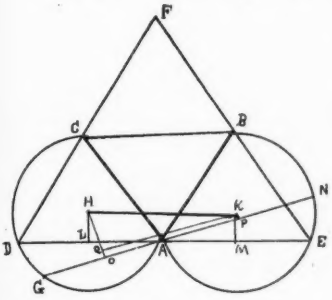
the same value as found above for x. Hence the expressions are equal when n and m are indefinitely increased.

279. Proposed by C. C. WENTWORTH, C. E., Roanoke, Va.

To construct geometrically the maximum equilateral triangle circumscribed about a given triangle.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC be the given triangle. On AC, AB describe segments to contain angles of 60°. Let H, K be the centers of these circles. Join HK and parallel to HK draw DAE. Join DC and EB, and produce DC, EB until they meet in F. Then FDE is the required maximum equilateral triangle.



$$\angle D = \angle E = \angle F = 60°.$$

Draw any other line GN through A. Let fall the perpendiculars HL, KM, HO, KP, and draw KQ parallel to GN.

$$KQ = OP = \tfrac{1}{2}GN;\ HK = LM = \tfrac{1}{2}DE,\ HK > KQ. \quad \therefore DE > GN.$$

$\therefore DE$ is the longest line that can be drawn through A.

$\therefore FDE$ is the maximum equilateral triangle required.

The same method will serve to describe the maximum triangle having any angles, about ABC.

Also solved by J. Scheffer.

MISCELLANEOUS.

155. Proposed by A. H. HOLMES, Brunswick, Maine.

There are two vessels, one containing a gallons of alcohol, the other b gallons of water. Suppose that c gallons are simultaneously taken from each and poured into the other, how many times must this be done so that there will be the same proportion of alcohol to water in each vessel?

I. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let $F(x)$ represent the number of gallons in the first vessel after x operations, then the number of gallons in the first vessel after the next operation will be $F(x) - \frac{c}{a}F(x) + \frac{c}{b}[a - F(x)]$. $\quad \therefore F(x+1) - [1 - (\frac{c}{a} + \frac{c}{b})]F(x) = \frac{ac}{b}$.

By the calculus of finite differences, we get

$$F(x) = \frac{a^2}{a+b} + C[1 - (\frac{c}{a} + \frac{c}{b})]^x.$$

Since for $x=0$, $F(x)=a$, we get $C = \frac{ab}{a+b}$.

$$\therefore F(x) = \frac{a^2}{a+b} + \frac{ab}{a+b}[1 - (\frac{c}{a} + \frac{c}{b})]^x.$$

It is easily seen that when the same proportion of alcohol to water prevails, the contents of the alcohol in the first vessel will be $=a^2/(a+b)$.

$\therefore x$ must be $=\infty$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

After the first operation, there are $a-c$ gallons of alcohol in the first vessel, and c gallons of alcohol in the second vessel. After the second operation, there are $a-2c+c^2(1/a+1/b)$ gallons of alcohol in the first vessel, and $2c-c^2(1/a+1/b)$ gallons of alcohol in the second vessel.

Let $A=c(1/a+1/b)$. Then, after the third operation, there are $a-3c+3Ac-A^2c$ gallons of alcohol in the first vessel, and $3c-3Ac+A^2c$ gallons of alcohol in the second vessel. After the nth operation there are

$$a-nc+\frac{n(n-1)}{2!}Ac-\frac{n(n-1)(n-2)}{3!}A^2c+\ldots\ldots\pm A^{n-1}c=a+\frac{c(1-A)^n-c}{A}$$ gallons of alcohol, and $\frac{c-c(1-A)^n}{A}$ gallons of water in the first vessel, and

$$nc-\frac{n(n-1)}{2!}Ac+\frac{n(n-1)(n-2)}{3!}A^2c-\ldots\ldots\pm A^{n-1}c=\frac{c-c(1-A)^n}{A}$$ gallons of alcohol, and $b+\frac{c(1-A)^n-c}{A}$ gallons of water in the second vessel.

$$\therefore \frac{Aa+c(1-A)^n-c}{c-c(1-A)^n}=\frac{c-c(1-A)^n}{Ab+c(1-A)^n-c}.$$

$$\therefore (1-A)^n=\frac{c(a+b)-Aab}{c(a-b)}=0. \quad \therefore n=-\infty, \text{ or } A=1.$$

$\therefore$ The result stated can only happen when $a=b=2c$, then $n=1$.

156. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

There exist no multiply perfect odd numbers of multiplicity n containing only n distinct primes.

Solution by JACOB WESTLUND, Ph. D., Purdue University, Lafayette, Ind.

If n denotes the multiplicity of a multiply perfect number $p_1^{a_1}p_2^{a_2}\ldots\ldots p_i^{a_i}$, where $p_1, p_2, \ldots\ldots$ are distinct primes, we have

$$n=\frac{p_1-\frac{1}{p_1^{a_1}}}{p_1-1}\cdot\frac{p_2-\frac{1}{p_2^{a_2}}}{p_2-1}\ldots\ldots, \text{ and hence } n<\frac{p_1}{p_1-1}\cdot\frac{p_2}{p_2-1}\ldots\ldots\frac{p_i}{p_i-1}.$$

Now, if $p_1>2$, we have

$$\frac{p_1}{p_1-1}\cdot\frac{p_2}{p_2-1}\cdots\cdots\frac{p_i}{p_i-1}\leq\frac{3}{2}\cdot\frac{4}{3}\cdots\cdots\frac{i+2}{i}=\frac{i+2}{2}.$$

Hence we should have $n<\frac{i+2}{2}$, or $i>2n-1$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

256. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Three men, A, B, and C, rented a pasture for a fixed amount, each to pay per month in proportion to the stock pastured. During the first month A put in 3 horses and B and C each some horses, and B paid for the month \$6, but A and C each defaulted payment. During the next month each put in one more horse, and C paid for the month \$7.20, but A and B each defaulted payment. During the next month each put in one more horse, and A paid his bill for the month, \$5, but B and C each defaulted.

Required: (1) the rent of the pasture per month; (2) the number of horses B and C each put in during the first month; and (3) the amount A, B, and C, each, owed for the unpaid service.

257. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Solve (1) $x+y=10$, (2) $3x=\log_{10}y$.

258. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Sum the infinite series $\frac{n^2}{(4n^2-1)^2}$ beginning with $n-1$, n being always odd.

CALCULUS.

216. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Find the limit of the sum of the series

$$\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots\cdots+\frac{n}{n^2+m^2},$$

when n and m are indefinitely increased. (Distinguish the several cases arising from the different *relative* values of m and n.)

217. Proposed by S. A. COREY, Hiteman, Iowa.

In *The Analyst*, Vol. II, p. 120, 1875, G. W. Hill finds by the method of mechanical quadrature the value of $\int_0^{\frac{1}{3}\pi}\frac{xdx}{\sin x[1+.16\cos^2x]^{\frac{1}{2}}}$ to be 1.6576363.

Evaluate the definite integral by some other method and verify above result.

DIOPHANTINE ANALYSIS.

132. Proposed by DR. OSWALD VEBLEN, Princeton University, Princeton, N. J.

From the numbers, 0, 1, 2, ..., 42, select seven, such that the 42 differences of these seven numbers shall be congruent (mod. 43) to the numbers 0, 1, 2, ..., 42. The differences may be both + and −.

133. Proposed by R. D. CARMICHAEL, Hartselle, Alabama.

Find all perfect numbers of four primes and of multiplicity 4.

GROUP THEORY.

14. Proposed by O. E. GLENN, Springfield, Mo.

Hölder has proved* that any group (G) of order $\sum_{i=1}^{n} p_i$ (p_i a prime $\neq p_j$) may be generated as follows: $M^\mu = N^\nu = 1$, $N^{-1}MN = M^\alpha$, where $\{M\}$ is the product of all the invariant subgroups of G of prime order and $\{N\}$ is any one of a set of conjugate cyclical subgroups of order ν, ($\sum_{i=1}^{n} p_i = \mu\nu$). Find the generating relations of G in terms of operations of prime order, and express M and N in terms of these operations, for $n=4$.

MECHANICS.

187. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Find the path described by a particle acted upon by a central force, the force being directly proportional to the distance of the particle.

188. Proposed by H. L. ORCHARD, M. A., B. Sc. (Unsolved problem in the Educational Times, London.)

Spherical bubbles are rising in water. Find the relation between radius and velocity.

UNSOLVED PROBLEMS.

NOTE. The following problems still remain unsolved (in our columns).

Calculus, 209. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A thread makes $n(=30)$ equidistant spiral turns around a rough cone whose altitude is $h(=10$ feet), and radius of base $r(=11$ inches). How far will a bird fly in unwinding the thread if the part unwound is at all times perpendicular to the axis of the cone?

Calculus, 212. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Show that any root of the equation $y^5 - 5y = 4x$ satisfies the differential equation $\frac{d^4y}{dx^4} = (\frac{4}{5})^4 x^{-3} \frac{d^4(x^{-\frac{1}{4}}y)}{d(x^{-\frac{1}{4}})^4}$. Generalize the problem.

*See Burnside, *Theory of Groups*, p. 353.

NOTES AND NEWS.

At Syracuse University, Professor O. S. Stetson has been promoted to an assistant professorship of mathematics.

Popular Science Monthly of February contains an article by Professor G. A. Miller on "Some recent tendencies in mathematical instruction."

At Rutgers College, Mr. Richard Morris has been promoted from an instructorship to an associate professorship in graphics and applied mathematics.

Mr. Charles Haseman, assistant in mathematics at Indiana University, has obtained leave of absence and will spend the summer in graduate study at Göttingen.

Bowdoin College has received from Colonel I. H. Wing of Batfield, Wisconsin, a fund of $50,000, to endow the chair of mathematics. Colonel Wing is an alumnus of Bowdoin.

Professor C. A. VanVelzer, head professor of mathematics at the University of Wisconsin, has resigned, the resignation to take effect July 1, 1906. He intends to retire from teaching.

The San Francisco Section of The American Mathematical Society met at Stanford University on February 24. The printed program announced fifteen papers to be read at the meeting.

Mr. H. W. Reddick, a graduate of Indiana University, and until February first, fellow in the University of Illinois, has been appointed to a position in Shortridge High School at Indianapolis.

A bill granting permission to Professor Simon Newcomb, U. S. N., to accept the decoration of the order 'Pour le Mérite, für Wissenschaften und Kunste,' tendered by the emperor of Germany, passed the senate on February 8.

Professor H. Poincare's address, "The Present and Future of Mathematical Physics," delivered before the section of applied mathematics at the St. Louis International Congress of 1904, is published in the February *Bulletin of the American Mathematical Society*. The translation is by Professor J. W. Young.

We learn from the *Bulletin of the American Mathematical Society* that a memorial to the late Dr. George Salmon, Provost of Trinity College, Dublin, was unveiled Friday, January 5, in the National Cathedral of St. Patrick's, Dublin. It consists of two windows in one of the chapels, a portrait of Dr. Salmon, and a Latin inscription bearing testimony to his mathematical and theological work.

Dr. W. B. Smith, professor of mathematics, Tulane University, is an author of wide reputation in non-mathematical fields. His latest book, "The Color Line," is reviewed in the July (1905) *Monist* by Professor C. J. Keyser. Says the reviewer:

"The book in hand is the first of its kind to be written by a mathematician; and all the qualities of the mathematical mind, excepting that of proverbial dryness, are evident throughout, in its grasp and penetration, in the clearness and steadiness of its vision, in the sharp precision with which its problems are stated, and in the boldness, energy, and relentless logical rigor with which they are handled. * * * From beginning to end the appeal is from the individual standard to the race standard; from the traditional maxims however kindly, to the warning dictates of science however stern and cold; from the relative impotence of education to the 'omniprepotence of heredity.' * * * Whether one does or does not agree with Professor Smith's conclusions, the candid reader will allow the book is one with which future discussions of its difficult problems will be compelled to reckon."

WASHINGTON MATHEMATICS TEACHERS.

The regular annual meeting of the Association of Teachers of Mathematics in Washington was held on December 29 at North Yakima. The following program was rendered at the meeting:

"Preliminary report on a proposed text of high school mathematics," by Professor Robert E. Moritz, University of Washington; "Present day defects in the teaching of secondary mathematics," by Mr. J. C. Keith, Seattle High School; "The relation of mathematics to applied science," by Professor O. L. Waller, Washington State College; "A comparison of some definitions with reference to the derivation of operations from them," by Professor W. A. Bratton, Whitman College; "Mathematics in the high school: Its subject matter and methods," by Mr. J. L. Dunn, Spokane High School

The following officers were elected for the ensuing year: President, Professor Robert E. Moritz, University of Washington; Vice President, Mr. J. C. Keith, Seattle High School; Secretary-Treasurer, Miss Zella E. Bisbee, North Yakima High School.

PORTRAITS OF EMINENT MATHEMATICIANS.

Part II of the Portfolio of Portraits of Eminent Mathematicians, which is edited by Professor David Eugene Smith, has been issued by the Open Court Publishing Company of Chicago. The installment forms a beautiful collection of pictures, if possible surpassing in interest and value the notable collection forming Part I. The set includes twelve portraits on imperial Japanese vellum, 11 x 14, each portrait being accompanied by a brief biographical sketch, with occasional notes of interest concerning the artists represented. The collection includes: Pierre Simon Laplace, from a painting by Naigeon; Blaise Pascal, after an etching; Guillaume de l' Hopital, from a painting by Foucher; Niccolo Tartaglia, from an old engraving; Jakob Bernoulli, after an etching by Dupin; Gaspard Monge, from a lithograph by Delpech; Joseph Louis Lagrange, after an

engraving by Hart; Bonaventura Cavalieri, after a drawing by Alfieri; Johann Bernoulli, from a painting by Rüber; Leonhard Euler, from a painting by Darbes; Isaac Barrow, from the statue by Noble; Carl Frederich Gauss, from a painting by Jensen.

This series of portraits should adorn the walls of every room where mathematics is taught.

CURRENT PUBLICATIONS.

The opening (January) number of the *Transactions of the American Mathematical Society* contains the following contributions: "On the relation between the three parameter groups of a cubic space curve and a quadric surface," by A. B. Coble; "On certain hyperabelian functions which are expressible by theta series," by J. I. Hutchinson; "On the form of a plane quintic curve with five cusps," by P. Field; "The symbolic treatment of differential geometry," by A. W. Smith; "Groups whose orders are powers of a prime," by W. B. Fite; "Differential parameters of the first order," by H. Maschke; "The Kronecker-Gaussian Curvature of hyperspace," by H. Maschke; "Groups containing only three operators which are squares," by G. A. Miller; "Theorems converse to Riemann's on differential equations," by D. R. Curtiss; "General mean value and remainder theorems," by G. D. Birkhoff; "Determination of the abstract groups of order p^2qr, p, q, r being distinct primes," by O. E. Glenn; "Note on the differential invariants of a surface and of space," by C. N. Haskins; "On improper multiple integrals," by James Pierpont.

The current number of the *American Journal of Mathematics* contains, "On the quaternary linear homogeneous groups modulo p of order a multiple of p," by L. E. Dickson; "On the integration of a system of differential equations in kinematics," by J. Eiesland; "On the determination of the properties of the nodal curve of a unicursal ruled surface," by C. H. Sisam; "Certain surfaces with plane or spherical lines of curvature," by L. P. Eisenhart; "The motion of a solid in an infinite liquid," by A. G. Greenhill.

The January number of the *Annals of Mathematics* contains the following papers: "The harmonic analysis of the semicircle and of the ellipse," by A. E. Kennelly; "Note on the possible number of operators of order 2 in a group of order 2^m," by G. A. Miller; "A geometrical problem connected with the continuation of a power series," by H. Maschke; "On the determination of a catenary with given directrix and passing through two given points," by H. F. MacNeish; "Concerning the discontinuous solution in the problem of the minimum surface of revolution," by H. F. MacNeish; "Introduction to the theory of Fourier series," by M. Bôcher.

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The Paris Academy of Sciences announces the Gutzman Prize of 100,000 francs for communication with any star or planet other than Mars. No affront to the Martians is intended.

Mr. John A. Bryan of Forney, Texas, has published a little booklet called *Perplex Problems*. It is a compilation of arithmetic, algebra, trigonometry, and miscellaneous problems, "many of which" the author says are "odds and ends and old puzzlers." The book has 38 pages, including six pages of answers, and the price is ten cents. The following are specimens:

"From a 50 gallon cask full of wine 10 gallons are drawn. The cask is then refilled with water. This is done ten times: how many gallons of wine remain?"

"A house and barn are 20 rods apart; the house is 10 and the barn 6 rods from a straight brook; what is the length of the shortest path by which one can go from the house to the brook and take water to the barn?"

The curious French and German poetical tributes to Archimedes, quoted in our November issue have called forth the following:

Now I, even I, would celebrate
In rhymes inapt, the great
Immortal Syracusan, rivaled nevermore,
Who in his wondrous lore,
Passed on before,
Left men his guidance how to circles mensurate.

[Adam C. Orr in the *Literary Digest.*]

Now, O hero, a great advancing in method,
Which you would proclaim wonderful, worketh universal;
Yet in our memories your labour is rooted;
Unto the end shouldst thou be amongst immortals!

[R. D. Carmichael].

ERRATUM.

Page 21, line 21, for $V_n = a' T^{n-1} \frac{\sin n\theta}{\sin\theta}$, read $U_n =$ etc.

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CERTAIN FACTORS OF THE GROUP DETERMINANT.

By WILLIAM BENJAMIN FITE.

So far as I know the following method for finding factors of a group determinant is new.

To the n operators $s_1, s_2, \ldots, s_n$ of a group G of finite order we make correspond n independent variables $x_{s_1}, x_{s_2}, \ldots, x_{s_n}$, such that $x_{s_k} \equiv x_{s_i s_j}$ if $s_k = s_i s_j$. With these variables we form the following determinant, which is called the group determinant of G:

$$\Theta(x) \equiv \begin{vmatrix} x_{s_1 s_1^{-1}} & x_{s_1 s_2^{-1}} & \cdots & x_{s_1 s_n^{-1}} \\ x_{s_2 s_1^{-1}} & x_{s_2 s_2^{-1}} & \cdots & x_{s_2 s_n^{-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ x_{s_n s_1^{-1}} & x_{s_n s_2^{-1}} & \cdots & x_{s_n s_n^{-1}} \end{vmatrix}.$$

If we take for G the symmetric group of degree 3 and put $s_1 \equiv 1$, $s_2 \equiv (abc)$, $s_3 \equiv (acb)$, $s_4 \equiv (ab)$, $s_5 \equiv (bc)$, $s_6 \equiv (ac)$, then

$$\Theta(x) \equiv \begin{vmatrix} x_{s_1} & x_{s_3} & x_{s_2} & x_{s_4} & x_{s_5} & x_{s_6} \\ x_{s_2} & x_{s_1} & x_{s_3} & x_{s_5} & x_{s_6} & x_{s_4} \\ x_{s_3} & x_{s_2} & x_{s_1} & x_{s_6} & x_{s_4} & x_{s_5} \\ x_{s_4} & x_{s_6} & x_{s_5} & x_{s_1} & x_{s_3} & x_{s_2} \\ x_{s_5} & x_{s_4} & x_{s_6} & x_{s_2} & x_{s_1} & x_{s_3} \\ x_{s_6} & x_{s_5} & x_{s_4} & x_{s_3} & x_{s_2} & x_{s_1} \end{vmatrix}.$$